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Preface

The present book originated as lecture notes of our courses in different fields of mechanics and engineering, revealing that typical master students either completely forget or do not know some of the basic concepts of higher mathematics that are needed for proper understanding the specific material in mechanics. Depending on the nature of the course and the student average level in mathematics, we had to devote several lectures just to cover students' shortage knowledge in mathematics.

The decision to write a lecture course on specific topics of higher mathematics that are admittedly indispensable for master students' was supported by our colleagues from the Institute for Problems in Mechanics of Russian Academy of Sciences (Moscow, Russia) and INSA de Lyon (Lyon, France).

The book is divided into chapters covering topics on topology, metric, normed, and functional spaces. We wrote a brief introduction to the theory of distributions, elements of complex analysis, and several sections on wavelet approximations. There is also a chapter on integral transforms including Fourier, Laplace, Mellin, and some other integral and discrete transforms. We wrote a rather detailed exposition of the theory of matrices, including functions of matrices and a special but important case of non-semisimple degeneracy. The course also contains a chapter on ordinary differential equations, including introduction to Hamiltonian formalism and a survey of the relevant numerical methods.

The authors are acknowledged to all our colleagues and students who helped us in preparing the lecture course and the manuscript.

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Chapter 1. Equation Chapter 1 Section 0 Topological, metric, functional, and vector spaces

This chapter presents the basic mathematical facts and concepts needed for the theory of vibrations, namely: basic properties of the elementary functions, complex variable method, methods of linear algebra, and some basic facts of the theory of ordinary differential equations. The reader familiar with these topics can easily pass to the subsequent chapters.

1.1. Equation Chapter 1 Section 1 Basics of topological and metric spaces

This paragraph is devoted to The main reference books for this chapter are Bourbaki (1989, 1998), Edwards (1995), and Hörmander (2003).

1.1.1. Topological spaces

Definition 1.1.1 (Topological space)

Topological space T is a space containing a set Λ of its subsets (called topology of the space T) with the following properties:

I. $\varnothing \in \Lambda$;

II. If $L_1, L_2 \in \Lambda$, then $L_1 \cap L_2 \in \Lambda$ and $L_1 \cup L_2 \in \Lambda$;

III. $\bigcup_{L \in \Lambda} L = T$ (union of all the subsets L belonging to Λ coincides with T).

Subsets L are called “open” sets.

Definition 1.1.2 (Open vicinity**)**

An open vicinity of the point $x \in T$ is any subset $L \in \Lambda$, containing x.

Definition 1.1.3 (Open subset)

A subset $C \subset T$ is called open, if it belongs to the open set Λ.

Definition 1.1.4 (Closed subset)

A subset $C \subset T$ is called closed, if it is a complement to an open set.

Definition 1.1.5 (Everywhere dense subset)

A subset $S \subset T$ is called everywhere dense, if any open vicinity $L \in \Lambda$ contains at least one point from S.

Definition 1.1.6 (Separable topological space)

Topological space is called (closed) separable, if it contains a countable everywhere dense subset.

Definition 1.1.7 (Homeomorphism)

Let X, Y be two topological spaces.

I. A map $f: X \to Y$ is called continuous, if $f^{-1}(V) \subset X$ is open for any open $V \subset Y$.
II. A map $f: X \to Y$ is called homeomorphism if f is a one-to-one correspondence and both f and f^{-1} are continuous functions.

Remark 1.1.1 (Locally convex topological space)

In the subsequent analyses all the topological spaces will be assumed to be *locally convex*. This means that their topologies can be defined by the corresponding sets of the *convex* subsets.

1.1.2. Metric and normed spaces

Definition 1.1.8 (Metric space)

Topological space T is called a metric space, if its topology is defined by a distance function $d: T \times T \to \mathbb{R}$ with the following properties:

I. $d(x, y) = 0$, if and only if $x = y$
II. $d(x, z) \leq d(x, y) + d(y, z)$ (inequality of triangle)

Topology in the metric space is defined by a system of balls $V_{x,\delta}$ of radius $\delta > 0$ with origins at points $x \in T$:

$$\underset{x\in T}{\forall x}\ \underset{\delta>0}{\forall \delta}\quad y \in V_{x,\delta} \Leftrightarrow d(x,y) < \delta \tag{1.1.1}$$

Inequalities I and II from the preceding definition imply:

Proposition 1.1.1

$$d(x,y) \geq 0, \qquad d(x,y) = d(y,x) \tag{1.1.2}$$

Definition 1.1.9 (Normed space)

Topological *vector* space T is called a normed space, if its topology is defined by a norm $N : T \to \mathbb{R}$ with the following properties:

$$N(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0\,, \tag{1.1.3}$$

$$N(\mathbf{x}+\mathbf{y}) \leq N(\mathbf{x}) + N(\mathbf{y}) \quad \underset{\mathbf{x},\mathbf{y}\in T}{\forall \mathbf{x},\mathbf{y}}\,, \tag{1.1.4}$$

$$N(t\mathbf{x}) = |t|\, N(\mathbf{x}) \quad \underset{t\in\mathbb{C}}{\forall t}\ \ \underset{\mathbf{x}\in T}{\forall \mathbf{x}}\,. \tag{1.1.5}$$

It can be shown that conditions (1.1.3) – (1.1.5) ensure:

Proposition 1.1.2

$$\underset{\mathbf{x}\in T}{\forall \mathbf{x}}\quad N(\mathbf{x}) > 0\,. \tag{1.1.6}$$

Definition 1.1.10 (Cauchy sequence)

Cauchy sequence is an infinte countable sequence $\{\mathbf{x}_n\}$, whose elements become infinitely close with the incresing number, this means

$$\underset{\varepsilon>0}{\forall \varepsilon}\ \exists n_\varepsilon\ \underset{n>n_\varepsilon.}{\forall n}\ \ N(\mathbf{x}_{n_\varepsilon} - \mathbf{x}_n) < \varepsilon \tag{1.1.7}$$

Definition 1.1.11 (Complete normed space)

Banach space is a *complete* normed space, which means that any Cauchy sequence converges to an element belonging to this space.

Remark 1.1.2

The norm of the Banach space is quite often denoted by $\|\mathbf{x}\|$ or $|\mathbf{x}|$.

Example 1.1.1 (L^p-norm in a finite dimensional vector space with finite $p \in \mathbb{R},\ p \geq 1$)

Let T be n-dimensional vector space, then L^p-norm (denoted by $\|\cdot\|_{L^p}$) is a function

$$\underset{\mathbf{x}\in T}{\forall \mathbf{x}} \quad \|\mathbf{x}\|_{L^p} \equiv \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, \tag{1.1.8}$$

where $x_k,\ k = 1,...,n$ are coordinates of vector $\mathbf{x}$ in a particular basis. Direct verification shows that conditions (1.1.3) – (1.1.5) are satisfied at any $p \geq 1$.

Remark 1.1.3 (Euclidian norm)

The L^2-norm (1.1.8) at $p = 2$ is used most often:

$$\underset{\mathbf{x}\in T}{\forall \mathbf{x}} \quad \|\mathbf{x}\|_{L^2} \equiv \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \equiv \left(\mathbf{x}\cdot\bar{\mathbf{x}}\right)^{1/2} \equiv |\mathbf{x}| \tag{1.1.9}$$

This norm is sometimes called as Euclidian norm.

Example 1.1.2 (L^∞-norm in a finite dimensional vector space)

Such a norm is defined by

$$\|\mathbf{x}\|_{L^\infty} \equiv \max_k |x_k|. \tag{1.1.10}$$

L^∞-norm is sometimes called uniform norm. It can be shown, that conditions (1.1.3) – (1.1.5) are satisfied.

Example 1.1.3 (L^p-norm in an infinite dimensional vector space at finite $p \in \mathbb{R},\ p \geq 1$)

Let T be a vector space of the all integrable on some set X functions, then L^p-norm (denoted by $\|\cdot\|_{L^p}$) is a map $T \to \mathbb{R}_+$ defined by

$$\underset{f \in T}{\forall f} \qquad \|f\|_{L^p} \equiv \left(\int_X |f(x)|^p \, dx \right)^{1/p}. \tag{1.1.11}$$

Example 1.1.4 (L^∞-norm in an infinite dimensional vector space of numerical functions**)**

$$\|f\|_{L^\infty} \equiv \sup_{x \in X} |f(x)|. \tag{1.1.12}$$

Such a norm is called the uniform norm.

Remark 1.1.4

A. Condition $p \geq 1$ in Examples 1.1.1 and 1.1.3 is needed to satisfy inequality (1.1.4), known also as Minkowski inequality. At $p < 1$ condition (1.1.4) fails.

B. In any of functional spaces L^p, $1 \leq p \leq \infty$, space of continuous functions is dense in the corresponding L^p-topology.

C. The following embedding of spaces L^p takes place:

$$L^q \subset L^p, \qquad q > p, \tag{1.1.13}$$

and at $q > p$ the topology L^q is stronger than topology L^p induced in L^q.

Theorem 1.1.1 (Hölder's inequality)

A. Let f be an integrable function, then a set I of real p, $1 \leq p \leq \infty$, at which L^p-norms $\|f\|_{L^p}$ are finite, is either empty, or a closed interval. In the latter case $\log(\|f\|_{L^p})$ is a convex function of $1/p$.

B. If integrable function f has a finite support, then the interval I is either empty, or has $p = 1$ as the starting point. In the latter case $\|f\|_{L^p}$ is the increasing function of p.

C. Let $f \in L^P$ and $g \in L^q$, where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1.1.14}$$

then $fg \in L^1$ and

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q} \tag{1.1.15}$$

Remark 1.1.5

Inequality (1.1.15) is known as Hölder's inequality. At $p = 2$ (and hence $q = 2$) inequality (1.1.15) is also called as Cauchy – Bounjakowsky inequality.

1.1.3. Hilbert spaces

Definition 1.1.12 (Hilbert space)

Function g mapping *real* vector space T into $\mathbb{R}$, or into $\mathbb{C}$, if T is a *complex* vector space, is called linear, if

$$\begin{gathered} \underset{\mathbf{x},\mathbf{y}\in T}{\forall \mathbf{x},\mathbf{y}}\ g(\mathbf{x}+\mathbf{y}) = g(\mathbf{x}) + g(\mathbf{y}) \\ . \\ \underset{\mathbf{x}\in T}{\forall \mathbf{x}} \ \underset{c\in\mathbb{R} \text{ or } \mathbb{C}}{\forall c}\ g(c\mathbf{x}) = cg(\mathbf{x}) \end{gathered} \tag{1.1.16}$$

Function satisfying conditions (1.1.16) is quite often called *linear form*.

Definition 1.1.13 (Bilinear form)

Function g mapping *real* vector space $T \times T$ into $\mathbb{R}$, is called *bilinear*, if it is linear with respect to each of its arguments. Such a function is quite often called *bilinear form*.

Definition 1.1.14 (Symmetric bilinear form)

Bilinear form g is called *symmetric*, if

$$\underset{\mathbf{x},\mathbf{y}\in T}{\forall \mathbf{x},\mathbf{y}} \quad g(\mathbf{x},\mathbf{y}) = g(\mathbf{y},\mathbf{x}) \quad . \tag{1.1.17}$$

Definition 1.1.15 (Sesquilinear form)

Function g mapping *complex* vector space $T \times T$ into $\mathbb{C}$, is called *sesquilinear*, if it is linear with respect to the first argument at fixed second argument, and the *complex-conjugate* function $\overline{g}$ (this will be precisely defined later on in this chapter) is linear with respect to its second argument at fixed first argument. Such a function is quite often called a *sesquilinear form*.

Definition 1.1.16 (Hermitian form)

Bilinear form g is called *Hermitian*, if

$$g(\mathbf{x},\mathbf{y}) = \overline{g(\mathbf{y},\mathbf{x})}\,. \tag{1.1.18}$$

Definition 1.1.17 (Scalar product)

Symmetric or Hermitian form (depending upon real or complex vector space is considered) is called *scalar* product, if

$$\underset{\mathbf{x}\in T,\ \mathbf{x}\neq 0}{\forall \mathbf{x}} \quad g(\mathbf{x},\mathbf{x}) > 0\,. \tag{1.1.19}$$

Quite often scalar product is denoted by $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$.

Definition 1.1.18 (Hilbert space)

A normed space (usually Banach space) is called Hilbert space, if it is complete and its norm is defined by a scalar product

$$\|\cdot\| \equiv \sqrt{\langle\cdot,\cdot\rangle}\,. \tag{1.1.20}$$

Remark 1.1.6 (Euclidian space)

A *finite* dimensional vector space equipped with the scalar product is called *Euclidean* space.

Proposition 1.1.3 (Cauchy –Schwartz inequality, known also as the Cauchy – Bunyakovsky – Schwartz inequality)

$$\underset{\mathbf{x},\mathbf{y}\in T}{\forall \mathbf{x},\mathbf{y}} \qquad \langle\mathbf{x},\mathbf{y}\rangle \le \sqrt{\langle\mathbf{x},\mathbf{x}\rangle}\sqrt{\langle\mathbf{y},\mathbf{y}\rangle} = \|\mathbf{x}\|\|\mathbf{y}\|\,. \tag{1.1.21}$$

Proof

The proof directly follows from inequality (1.1.19), yielding

$$\underset{\mathbf{x},\mathbf{y}\in T}{\forall \mathbf{x},\mathbf{y}} \quad \underset{c\in\mathbb{R}\text{ or }\mathbb{C}}{\forall c} \quad \langle\mathbf{x}-c\mathbf{y},\ \mathbf{x}-c\mathbf{y}\rangle \ge 0\,. \tag{1.1.22}$$

Taking

$$c = \frac{\langle \mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}, \tag{1.1.23}$$

where we assumed that $\langle \mathbf{y}, \mathbf{y} \rangle \neq 0$ (otherwise inequality (1.1.21) becomes trivial), and substituting (1.1.23) into (1.1.22), we arrive at the desired inequality (1.1.21).

Remark 1.1.7

Banach space L^p introduced in Example 1.1.3, at $p = 2$ can be associated with the corresponding Hilbert space, if scalar multiplication is defined by

$$\langle f, g \rangle \equiv \int_X f(x)\overline{g(x)}dx . \tag{1.1.24}$$

1.1.4. Duality

Definition 1.1.19 (Dual topological space)

Let $\Im$ be a topological space, then the dual space $\Im'$ is a space of all continuous linear forms (i.e. continuous linear functions defined on $\Im$ and taking values in $\mathbb{R}$).

Definition 1.1.20 (Weak topology)

It is possible to introduce the weakest topology in $\Im$, in which all the linear forms from $\Im'$ remain continuous. Such a topology is denoted by $\sigma(\Im, \Im')$, and it is called a *weakened topology* in $\Im$, since it is not stronger than the initial topology in $\Im$.

Similarly, in the dual space $\Im'$ (which can have no topology at all) it is possible to introduce a dual topology $\sigma(\Im', \Im)$, in which elements of $\Im$ regarded as linear forms, are continuous.

Definition 1.1.21 (Mackey topology)

In the initial topological space $\Im$ it is possible to introduce the strongest topology, in which all the linear forms from the dual space $\Im'$ remain continuous. Such a topology is known as Mackey topology, this is denoted by $\tau(\Im, \Im')$.

Remark 1.1.8

The initial topology T in $\Im$ satisfies the following condition:

$$\sigma(\Im,\Im') \leq T \leq \tau(\Im,\Im'), \tag{1.1.25}$$

where sign "$\leq$" means a weaker topology.

Proposition 1.1.4

Let real p $1 \leq p \leq \infty$ and q $1 \leq q \leq \infty$ satisfy relation (1.1.14), then topological spaces L^p and L^q of *numerical* functions are dual spaces.

1.1.5. Sobolev functional spaces

Definition 1.1.22 (Space of locally integrable functions; Sobolev space)

Let E be a topological space, and $C^k(E)$ be a set of all real (or complex) valued functions having continuous derivatives up to k-th order. This set is not complete in L^p_{loc}-topology $1 \leq p < \infty$ induced in $C^k(E)$, where L^p_{loc} is a space of all *locally integrable* in p-th power functions defined on E. The term locally integrable means that functions from L^p_{loc} are integrable on any bounded subsets of E. On $C^k(E)$ a stronger topology than L^p_{loc} can be defined by introducing the following semi-norms:

$$\|f\|^p_{k,\,loc} \equiv \sum_{m=0}^{k} \left\|f^{(m)}\right\|^p_{loc}, \tag{1.1.26}$$

where $f \in C^k(E)$. Even in this stronger topology the space $C^k(E)$ is not complete. Closer of $C^k(E)$ in topology defined by (1.1.26) is called *Sobolev space* and denoted by $W^p_{k,loc}$.

Remark 1.1.9

Quite often in applications topological space E is compact; for example, it can be a ball in $\mathbb{R}^n$ or a closed interval in $\mathbb{R}$: In such a case condition of local integrability can be substituted by condition of integrability, and the corresponding Sobolev space is denoted by W^p_k.

Proposition 1.1.5 (Sobolev embedding theorem)

Let E be a compact in $\mathbb{R}^n$ and k, m be natural numbers and $1 < p, q < \infty$. If

$$k > m \quad \& \quad k - n / p > m - n / q, \tag{1.1.27}$$

then

$$W_k^p \subseteq W_m^q \tag{1.1.28}$$

and embedding is continuous (W_k^p -topology is stronger than W_m^q).

Remarks 1.1.10 (Rellich-Kondrashov theorem)

A. Sobolev embedding theorem is also known as Rellich-Kondrashov theorem.

B. Conditions of the theorem remain valid if $q = \infty$. In such a case for any natural n and real p ($1 < p < \infty$) the following embedding theorem takes place:

$$W_k^p \subseteq C^m(E), \tag{1.1.29}$$

provided

$$k - n / p > m. \tag{1.1.30}$$

Thus, at satisfying condition (1.1.30) functions from W_k^p have continuous derivatives up to m -th order.

C. There are generalizations of Sobolev spaces with fractional index k; these generalizations known also as Hörmander spaces will be considered in Chapter 2.

1.2. Equation Chapter 1 Section 2 (Real) trigonometric, hyperbolic, and some other functions and series

In this section we present only those properties of the corresponding (real) elementary functions that will be needed for the further analyses. For references see Korn and Korn (2000) and Titchmarsh (1976).

1.2.1. Trigonometric functions

Definition 1.2.1 (Sine function; Cosine function))

Sine and cosine functions can be defined by the following equivalent equations.
1) These are functions defined by the following series:

$$\sin(x) \equiv \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \qquad (1.2.1)$$

$$\cos(x) \equiv 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!}. \qquad (1.2.2)$$

2) These functions are solutions of the following differential equation:

$$\left(\frac{d^2}{dx^2} + 1\right) f(x) = 0. \qquad (1.2.3)$$

Definition 1.2.2 (Real analytic function)

A function is (real) analytic (in particular vicinity), if it can be expanded into a power series, convergent in that vicinity.

Proposition 1.2.1

Power series in the right-hand sides of (1.2.1), (1.2.2) converge everywhere in $(-\infty, \infty)$, ensuring both sine and cosine to be (real) analytic functions

Proof

Proof of the proposition flows out directly from expressions (1.2.1), (1.2.2), and Stirling's estimate for the factorial; see:

$$n! \approx \sqrt{2\pi n}\, \exp\left(n\left(\log n - 1\right)\right), \quad n \to \infty. \qquad (1.2.4)$$

Combining (1.2.1), (1.2.2), (1.2.4) yields for sine and cosine functions:

$$\frac{x^k}{k!} \sim \frac{\exp\left(-k\left(\log k - 2\right)\right)}{\sqrt{2\pi k}}, \quad k \to \infty, \qquad (1.2.5)$$

where parameter k in (1.2.5) corresponds to $2n-1$ for sine and $2n$ for cosine function. Asymptotic estimate (1.2.5) ensures convergence of the regarded series.

Corollary

Any power series

$$\sum_k a_k \frac{x^k}{k!} \tag{1.2.6}$$

with coefficients a_k satisfying asymptotic estimate

$$|a_k| \sim o\left(\exp(k \log k)\sqrt{k}\right), \quad k \to \infty, \tag{1.2.7}$$

defines a real analytic function in $(-\infty, \infty)$. The symbol o, known as the small Landau symbol, denotes that sequence $|a_k|$ increases weaker at $k \to \infty$ than the expression in the right-hand side of (1.2.7).

Proof

Proof flows out from estimate (1.2.5).

Remark 1.2.1

From definitions (1.2.1) – (1.2.3) it can be difficult, if possible, to deduce that both sine and cosine are periodic functions.

We shall also need some other trigonometric functions, which definitions are given below.

Definition 1.2.3 (tangent function; Cotangent function)

Tangent and cotangent functions can be defined by the following equivalent equations.
1) These functions are expressed in terms of ratios of sine and cosine functions:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \tag{1.2.8}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)} \tag{1.2.9}$$

2) These functions are expressed in terms of power series (convergent at $|x| < \frac{\pi}{2}$ for tangent and $0 < x < \pi$ for cotangent):

$$\tan(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} x^{2k-1} \tag{1.2.10}$$

$$\cot(x) = \frac{1}{x}\sum_{k=1}^{\infty}(-1)^k \frac{2^{2k} B_{2k}}{(2k)!} x^{2k}, \tag{1.2.11}$$

where B_{2k} are Bernoulli numbers. These numbers can be calculated by the following recurrent formula:

$$B_0 = 1, \quad 1 + \binom{k}{1} B_1 + \binom{k}{2} B_2 + \ldots + \binom{k}{k-1} B_k = 0. \tag{1.2.12}$$

3) These functions are solutions of the following (nonlinear) differential equations:

$$\left(\frac{d^2}{dx^2} - \tan(x)\frac{d}{dx}\right)\tan(x) = 0, \tag{1.2.13}$$

$$\left(\frac{d^2}{dx^2} + \cot(x)\frac{d}{dx}\right)\cot(x) = 0. \tag{1.2.14}$$

Some basic properties of trigonometric functions:

$$\sin^2(x) + \cos^2(x) = 1, \tag{1.2.15}$$

$$\sin(2x) = 2\sin(x)\cos(x), \tag{1.2.16}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1, \tag{1.2.17}$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}, \tag{1.2.18}$$

$$\cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}, \tag{1.2.19}$$

$$\sin(A) \pm \sin(B) = 2\sin\frac{A \pm B}{2}\cos\frac{A \mp B}{2}, \tag{1.2.20}$$

$$\cos(A) + \cos(B) = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}, \tag{1.2.21}$$

$$\cos(A) - \cos(B) = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}, \tag{1.2.22}$$

$$\tan(A) \pm \tan(B) = \frac{\sin(A \pm B)}{\cos(A)\cos(B)}, \tag{1.2.23}$$